Free Vibrations of Noncircular Cylindrical Shell Segments

CARL EDWARD KURT*
McDonnell Douglas Astronautics Company East, St. Louis, Mo.

AND

DONALD EDWARD BOYD†
Oklahoma State University, Stillwater, Okla.

A method is developed to determine the natural frequencies and mode shapes of circular and noncircular cylindrical panels. The panels are assumed to be freely supported along their curved edges and to have arbitrary straight-edge boundary conditions. The Donnell equations of equilibrium for noncircular cylindrical shells are used. The displacement functions, assumed to be mixed power and trigonometric series, reduce the Donnell equations to a set of coupled, algebraic recurrence formulas. It is shown that this method yields the exact solution. A study was also made to investigate the effects of neglecting in-surface inertias. For each noncircular segment an equivalent circular panel, which vibrates at the same natural frequency as the corresponding noncircular panel, was found. A significant advantage of the power series method is that the natural frequencies of circular and noncircular panels can be obtained for all combinations of straight-edge boundary conditions without having to reformulate the problem for each set of boundary conditions.

Nomenclature

$A_{m,n},B_{m,n},C_{m,n}$ = displacement constants
a = radius of circular panel
D = bending rigidity
D_j = jth constant for nondimensional curvature
E = modulus of elasticity
G = shear modulus
h = thickness of shell
j = index of curvature constants
K = $Eh/(1 - \mu^2)$
k = number of terms necessary to express the curvature
L_x = length of shell in x direction
L_s = arc length of panel
M_ξ = moment stress resultant in ξ direction
m,n = indices on displacement summations
n_c = number of circumferential modes of a closed circular cylinder
p = maximum value of n for recurrence formulas
r = local radius of panel
t = time
V_ξ = effective transverse shear resultant
v_ξ = $\partial v/\partial \xi$
s,x,z = spatial coordinates
v,u,w = orthogonal displacements
$[X]$ = frequency matrix
α = $(1 + \mu)/2$
β_m = $m\pi L_s/L_x$
γ = $(1 - \mu)/2$
n
δ_j = 0 if $n < j$; = 1 if $n \geq j$
η = nondimensional x coordinate, x/L_x
θ_0 = subtended angle for circular panel
μ = Poisson's ratio
ξ = nondimensional s coordinate, s/L_s
ρ = mass density
ω = natural frequency, rad/sec
$\bar{\omega}^2$ = nondimensional natural frequency, $12(1-\mu^2)\rho L_s^2\omega^2/E$

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* Senior Engineer, Strength Department. Member AIAA.
† Associate Professor. Associate Fellow AIAA.

Introduction

THROUGH the years, the study of the free vibrations of circular cylindrical shells has received much attention because of the widespread industrial application of circular cylinders. Many times, however, designers are faced with the problem of designing cylindrical shells which are not circular. The purpose of this study is to investigate the free vibrations of noncircular cylindrical shell panels which have freely supported curved edges and arbitrary straight-edge boundary conditions.

Noncircular cylindrical shells are used in the products of many industries; for example, noncircular skin panels are often used on flight structure wing and tail surfaces, leading edges, and fuselages. Sometimes cylinders are designed to be circular but become noncircular during fabrication and should be analyzed as noncircular shells. There has also been an interest in submarine hulls with noncircular cross sections and noncircular skin panels between stiffeners. The roof structures of buildings may be unstiffened, noncircular cylindrical shells.

The geometry and nomenclature is shown in Fig. 1. The quantities are defined as follows: s, x, and z are the orthogonal coordinates; v, u, and w are corresponding displacement components; r is the variable radius of curvature; h is the shell thickness; L_x is the length of the shell in the x direction; L_s is the arc length of the cylindrical shell, measured along the surface in the s direction.

In 1894, Rayleigh,[1] assuming inextensional middle surfaces, derived an expression for the natural frequencies of closed circular cylinders with simply supported ends considering only the circumferential modes. The results from his expression agreed well with experimental data for cylinders vibrating in the higher circumferential modes. However, frequencies obtained by Rayleigh for the lower circumferential modes were in error. Using an energy approach and including middle surface extensions, Arnold and Warburton[2] clarified the significance of Rayleigh's inextensional assumption and explained the errors in the lower mode frequencies. Since their work, many papers have been published reporting parametric analytical studies and experimental investigations of simply supported closed circular cylinders.[3–5]

Stadler and Wang[6] obtained the natural frequencies of open circular panels. These panels were freely supported along their curved edges. They also satisfied the implied boundary conditions along the nodal lines of closed circular cylinders.

In one of the first attempts to study noncircular cylindrical shell problems, Marguerre,[7] in considering the stability of simply supported noncircular cylinders, expressed the cylinder curvature as an infinite Fourier cosine series. Malkina[8] assumed a nondimensional radius of curvature expressed as an infinite Fourier cosine series in his study of the free vibrations of noncircular cylindrical shells.

Kempner[9] verified the Donnell equations for noncircular cylindrical shells and studied static problems using a one-parameter special case of the infinite series assumed by Marguerre. Kempner's class of cylinders included doubly-symmetric ovals which, for low eccentricities, closely approximated elliptical cylinders.

Boyd[12] assumed the curvature expression as a finite power series. In order to nondimensionalize the curvature, the curvature was multiplied by the arc length L_s. The curvature was then assumed to be given by Eq. 1;

$$\frac{L_s}{r} = \sum_{j=1}^{k} D_j \left(\frac{s}{L_s}\right)^{j-1} \tag{1}$$

where r = radius of curvature, s = coordinate along the cross section, D_j = unitless constants dependent upon j, and k = number of terms necessary to express the curvature accurately. For the special case of a circular cross section with a constant radius, $r = a$, $k = 1$, and Eq. 1 reduces to

$$L_s/r = D_1 = \theta_0 a/a = \theta_0 \tag{2}$$

where θ_0 = the angle subtended by the circular panel.

Equation (1) was used in the present study because of the simplicity involved in deriving the frequency equation and because fewer terms may be required to describe the curvatures of some noncircular panels.

Formulation

Equations (3) are the familiar Donnell equations of equilibrium, modified for a freely vibrating noncircular cylindrical shell[10]:

$$u_{xx} + \frac{1-\mu}{2} u_{ss} + \frac{1+\mu}{2} v_{xs} + \mu \left(\frac{w}{r}\right)_x - \frac{(1-\mu^2)\rho}{E} u_{tt} = 0 \tag{3a}$$

$$\frac{1-\mu}{2} v_{xx} + v_{ss} + \frac{1+\mu}{2} u_{xs} + \left(\frac{w}{r}\right)_s - \frac{(1-\mu^2)\rho}{E} v_{tt} = 0 \tag{3b}$$

$$\frac{1}{r}\left[\frac{w}{r} + v_s + \mu u_x\right] + \frac{h^2}{12} \nabla^4 w + \frac{(1-\mu^2)\rho}{E} w_{tt} = 0 \tag{3c}$$

Subscripts indicate differentiation and x,s,z,t = longitudinal, circumferential, and transverse spatial coordinates, and time, respectively; u,v,w = displacements in the x,s,z directions, respectively; r = variable radius of curvature; μ = Poisson's ratio; ρ = mass density; and E = Young's modulus.

For convenience, Eqs. (3) are rewritten using the following nondimensional independent variables:

$$\eta = x/L_x,\ \xi = s/L_s,\ \alpha = (1+\mu)/2,\ \gamma = (1-\mu)/2$$

where L_x = length of cylinder in the x direction and L_s = arc length of cylinder in the s direction. Equations (3) then become

$$\left(\frac{L_s}{L_x}\right)^2 u_{\eta\eta} + \alpha u_{\xi\xi} + \left(\frac{L_s}{L_x}\right) v_{\eta\xi} + \mu \left(\frac{L_s}{L_x}\right)\left(\frac{L_s}{r} w\right)_\eta - \frac{(1-\mu)^2 \rho L_s^2}{E} u_{tt} = 0 \tag{4a}$$

$$\gamma\left(\frac{L_s}{L_x}\right)^2 v_{\eta\eta} + v_{\xi\xi} + \alpha\left(\frac{L_s}{L_x}\right) u_{\eta\xi} + \left(\frac{L_s}{r} w\right)_\xi - \frac{(1-\mu^2)\rho L_s^2}{E} v_{tt} = 0 \tag{4b}$$

$$\frac{L_s}{r}\left[\frac{L_s}{r} w + v_\xi + \mu\left(\frac{L_s}{L_x}\right) u_\eta\right] + \frac{h^2}{12} \bar{\nabla}^4 w + \frac{(1-\mu^2)\rho L_s^2}{E} w_{tt} = 0 \tag{4c}$$

where

$$\bar{\nabla}^4 w = \left(\frac{L_s}{L_x}\right)^2 \frac{1}{L_x^2} w_{\eta\eta\eta\eta} + \frac{2}{L_x^2} w_{\eta\eta\xi\xi} + \frac{1}{L_s^2} w_{\xi\xi\xi\xi}$$

Following the procedure of Boyd,[12] the solutions for the displacements were assumed to be mixed, doubly infinite series which satisfy freely supported boundary conditions at $x = 0$ and $x = L_x$. These displacement series are

$$u(\eta,\xi,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}\xi^{n-1} \cos m\pi\eta \cos\omega_m t \tag{5a}$$

$$v(\eta,\xi,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{m,n}\xi^{n-1} \sin m\pi\eta \cos\omega_m t \tag{5b}$$

$$w(\eta,\xi,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{m,n}\xi^{n-1} \sin m\pi\eta \cos\omega_m t \tag{5c}$$

where $A_{m,n}, B_{m,n}, C_{m,n}$ = unknown constant coefficients with units of length, $\xi = s/L_s$ and $\eta = x/L_x$.

The assumed displacement functions do not satisfy any specific boundary conditions along the straight edges (ξ = constant). The incorporation of arbitrary straight-edge boundary conditions into the problem will be discussed later in detail.

When the assumed displacement functions of Eqs. (5) are substituted into the equilibrium Eqs. (4), three coupled algebraic recurrence formulas are obtained, one for each equilibrium equation:

$$\left(-\beta_m + \frac{\bar{\omega}^2}{12\beta_m}\right) A_{m,n} + \frac{\gamma(n)(n+1)}{\beta_m} A_{m,n+2} + \frac{\alpha(n)}{\beta_m} B_{m,n+1} + \mu \sum_{j=1}^{k}\,^{n} D_j\delta_j C_{m,n-j+1} = 0 \tag{6a}$$

$$-\alpha A_{m,n+1} + \left(\frac{-\gamma\beta_m + \dfrac{\bar{\omega}^2}{12\beta_m}}{n}\right) B_{m,n} + \frac{n+1}{\beta_m} B_{m,n+2} + \sum_{j=1}^{k}\,^{n} D_j\delta_{j-1} C_{m,n+2-j} = 0 \tag{6b}$$

$$-\frac{12\mu\beta_m}{(n+2)(n+3)} \sum_{j=1}^{k}\,^{n} D_j\delta_j A_{m,n+1-j} + \frac{12}{(n+2)(n+3)} \sum_{j=1}^{k}\,^{n} D_j\delta_j (n-j+1) B_{m,n-j+2} + \frac{\left[\left(\dfrac{h}{L_s}\right)^2 \beta_m^4 - \bar{\omega}^2\right]}{(n+2)(n+3)} C_{m,n} - \frac{2(n)(n+1)}{(n+2)(n+3)} \beta_m^2 \times \left(\frac{h}{L_s}\right)^2 C_{m,n+2} + \left(\frac{h}{L_s}\right)^2 n(n+1) C_{m,n+4} + 12 \sum_{i=1}^{k}\,^{n} \sum_{j=1}^{k}\,^{n} D_i D_j \delta_i \delta_j C_{m,n+2-i-j} = 0 \tag{6c}$$

where

$$\beta_m = m\pi L_s/L_x,\ \bar{\omega}^2 = 12(1-\mu^2)\rho\omega^2{}_m L_s{}^2/E$$

$$\overset{n}{\delta_j} = 0 \text{ if } n < j;\ = 1 \text{ if } n \geq j$$

If it is desired to neglect the in-surface inertia terms in the formulation of the vibration problem, the terms containing $\bar{\omega}^2$ are omitted from the first two recurrence formulas.

The recurrence formulas are general enough that if eight unknown constants are found, the remainder of the unknown constants can be found through the use of the three recurrence formulas. These eight unknown constants are obtained from the boundary conditions.

The boundaries were chosen to correspond to $\xi = 0$ and 1, and the following boundary conditions were considered at $\xi = 0,1$:

$$\begin{aligned} &u = 0 \text{ or } \partial u/\partial\xi = 0, \quad v = 0 \text{ or } \partial v/\partial\xi = 0 \\ &w = 0 \text{ or } V_\xi = 0, \quad \partial w/\partial\xi = 0 \text{ or } M_\xi = 0 \end{aligned} \tag{7}$$

The displacement series were substituted into the aforementioned boundary conditions and evaluated at $\xi = 0$ and $\xi = 1$. Therefore, with the three recurrence formulas and the eight boundary conditions known, the natural frequencies and modes of free vibration could be found.

In the doubly infinite series expansion for the displacements of Eqs. (5), the m summation denotes the vibrational mode in the η direction. That is, the first mode in the η direction is the case for $m = 1$. The n summation includes an infinite number of constant coefficients for each m. If an infinite number of terms in the n summation is used, the assumed displacement functions will converge to the exact solution. Examination of the three recurrence formulas in Eqs. (6) reveals that the coefficients will approach zero as n becomes large. Thus, this method will give sufficiently accurate displacements and natural frequencies if the n summation is truncated when the omitted terms are negligible.

For each m, the recurrence formulas resulted in n-wise coupling of the constant coefficients of the displacement series. Because of this complexity, the constant coefficients of the recurrence formulas were not eliminated from the set of simultaneous equations.

The recurrence formulas and the eight boundary conditions were arranged into a frequency matrix. The frequency matrix, denoted as X, was arranged as shown in Eq. (8). The frequency equation is

$$\left[\begin{array}{c|c|c} BC(1) & & \\ BC(2) & & \\ \multicolumn{3}{c}{\text{Recurrence Formula (6a)}} \\ \hline & BC(3) & \\ & BC(4) & \\ \multicolumn{3}{c}{\text{Recurrence Formula (6b)}} \\ \hline & & BC(5) \\ & & BC(6) \\ & & BC(7) \\ & & BC(8) \\ \multicolumn{3}{c}{\text{Recurrence Formula (6c)}} \end{array}\right] \left\{\begin{array}{c} A_{m,1} \\ \cdot \\ A_{m,p+2} \\ \hline B_{m,1} \\ \cdot \\ \cdot \\ B_{m,p+2} \\ \hline C_{m,1} \\ \cdot \\ \cdot \\ C_{m,p+4} \end{array}\right\} = \{0\} \tag{8}$$

or

$$[X]\left\{\begin{array}{c} A \\ B \\ C \end{array}\right\} = \{0\} \tag{9}$$

where $BC(1)$ = boundary condition for u at $\xi = 0$, $BC(2)$ = boundary condition for u at $\xi = 1$, $BC(3)$ = boundary condition for v at $\xi = 0$, $BC(4)$ = boundary condition for v at $\xi = 1$, $BC(5)$ = first boundary condition for w at $\xi = 0$, $BC(6)$ = second boundary condition for w at $\xi = 0$, $BC(7)$ = first boundary condition for w at $\xi = 1$, and $BC(8)$ = second boundary condition for w at $\xi = 1$.

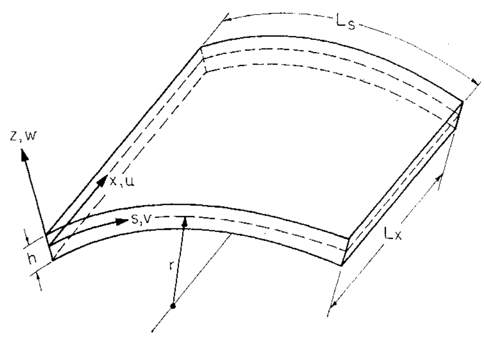


Fig. 1 Elemental geometry.

For each value of m, each recurrence formula was repeated p times. Because there were two boundary conditions for u and v, there were $(p + 2)$ coefficients, for $A_{m,n}$ and $B_{m,n}$. Using the same argument for w, there were $(p + 4)$ coefficients of $C_{m,n}$ in the frequency matrix. Therefore the frequency matrix is a $3p + 8$ square matrix.

It should be noted that the overall frequency matrix is not a convenient form. Because of the incorporation of the boundary conditions into the matrix, it is not of the usual eigenvalue form $|[X] - \omega^2[I]| = 0$. The rows containing the boundary conditions relate the unknown coefficients and are independent of the frequency $\bar{\omega}^2$. Because of this unconventional form, the methods which solve the standard eigenvalue problems could not be used.

Because Eq. (9) is homogeneous, the natural frequencies are those values of $\bar{\omega}^2$ which satisfy the condition that det $|X| = 0$. An iterative procedure was used to find the frequencies which satisfied the frequency criteria.

Theoretically, there are $3p$ natural frequencies which satisfy the frequency criteria when in-surface inertia terms are included. Unfortunately, only the lower frequencies could be obtained with any degree of accuracy. The higher frequencies become quite inaccurate and many may be imaginary. When the frequencies of the higher circumferential mode shapes are needed, a larger number of terms must be taken.

Once a natural frequency was calculated, the corresponding mode was found. As in the case for all free vibration problems, only the normalized modes can be obtained. The number of frequencies accurately calculated is related to the number of coefficients taken. The unknown coefficients of $A_{m,n}$, $B_{m,n}$, and $C_{m,n}$ were found as a function of the last $C_{m,n}$ coefficient. For convenience, this last coefficient was set equal to one.

Numerical Results

Because the displacement series converged slowly, a large number of terms were required to obtain the natural frequencies. Therefore, the procedure was programed for an IBM 360/50 digital computer.

One circular shell panel was studied in detail to verify the use of the power series method for calculating the natural frequencies of circular cylindrical shells. A second computer program was written, based on an exact solution for circular panels developed by Stadler and Wang,[6] to compare results with the frequencies obtained by the present method. In-surface inertia terms were included in this comparison. It was possible to use the same nondimensional shell parameters in both programs.

Table 1 Nondimensional natural frequencies $\bar{\omega}^2$ calculated by the power series method and Stadler and Wang's method[a]

n_c	$m = 1$		$m = 3$	
In-surface inertia terms included				
	Stadler and Wang	Power series	Stadler and Wang	Power series
2	0.38046	0.37644	6.4780	7.9342
3	0.04491	0.04504	1.4861	1.5853
4	0.00514	0.00515	0.4219	0.4241
5	0.00307	0.00308	0.1401	0.1416
6	0.00394	0.00394	0.0577	0.0579
7	0.00530	0.00531	0.0288	0.0295
8	0.00693	0.00694	0.0187	0.0193
9	0.00879	0.00880	0.0154	0.0159
In-surface inertia terms neglected				
2	0.48696	0.48168	7.3601	9.1141
3	0.05036	0.05045	1.5391	1.7497
4	0.00547	0.00548	0.4502	0.4525
5	0.00319	0.00320	0.1465	0.1479
6	0.00404	0.00405	0.0588	0.0597
7	0.00541	0.00542	0.0294	0.0302
8	0.00704	0.00704	0.0190	0.0196
9	0.00889	0.00890	0.0156	0.0161

[a] $h/L_x = 8.333 \times 10^{-4}$, $L_s/a = \pi/n_c$, $a/L_x = \frac{1}{4}$, $\mu = 0.30$, $P_{max} = 25$, $u = v_\xi = w = M_\xi = 0$.

The shell chosen for comparing the two methods had the following nondimensional properties:

$$L_s/L_x = 0.07854,\ h/L_s = 0.02546,\ L_s/r = 0.3927$$

$$\mu = 0.29,\ m = 1,\ u = v_\xi = w = M_\xi = 0,\ \text{at } \xi = 0,1$$

For these shell parameters, the value of p was varied. When each recurrence formula was repeated only a few times (a low value of p), the calculated natural frequencies were inaccurate. This implies that the neglected constant coefficients were significant. When the value of p was increased to 15, the lowest natural frequency differed by only 0.01% from those obtained by the trigonometric solution of Stadler and Wang. Thus, it was concluded that the power series method yielded the same solution as obtained by Stadler and Wang. The program was also capable of calculating the corresponding modes of free vibration.

To determine the range of panels which could be practically studied, the value of p was held at some practical limit and the nondimensional shell parameters were varied. The results of these computer runs were compared with results from the solution of Stadler and Wang to establish practical accuracy bounds on shell parameters. From the recurrence formulas of Eqs. (6), it is seen that the values of the constant coefficients depend upon the three geometric quantities, β_m, L_s/r, and h/L_s.

For efficient utilization of the computer, the maximum practical value of p for this study was 25. This value of p resulted in an 83 × 83 frequency matrix. For this study, the value of L_s/r for a circular panel was taken as

$$L_s/r = L_s/a = \pi/8,\ \beta_m = (m\pi^2/8)\cdot a/L_x$$

where a is the radius of panel.

Two different values of $h/L_s = 0.01273$ and 0.02546 ($h/a = 0.005$ and 0.01, respectively) were chosen. The m was set equal to 1, and the boundary conditions used were the same as those implied by the solution obtained by Stadler and Wang. They are $u = v_\xi = w = M_\xi = 0$ at $\xi = 0,1$.

The results of this study for $h/L_s = 0.02546$ showed that the accuracy of the nondimensional frequency $\bar{\omega}^2$ was within 15% for values of a/L_x equal to or less than 1.2. For $h/L_s = 0.01273$, $a/L_x = 1.2$, the calculated values of $\bar{\omega}^2$ were within 8% of actual values. Thus, for smaller values of h/L_s (thinner shells), the accuracy improved significantly. When frequencies for higher values of a/L_x are desired, the number of coefficients must be increased. Theoretically, the frequency will converge to the exact solution. In practice, computer round-off error may make the exact solution impossible to obtain. A similar case was studied with in-surface inertia terms neglected; similar results were obtained.

With the wide use of closed circular cylinders in industry, the extension of the results of this method to closed circular cylinders would be beneficial. When a closed circular cylinder vibrates, there are a number of nodal lines around the circumference. As an extension to this study, an equivalent panel for the closed circular cylinder was chosen between two adjacent nodal lines. The boundary conditions used for the equivalent panel were identical to the conditions existing along the nodal lines of the closed circular cylinder.

Table 1 illustrates the values of the nondimensional natural frequencies, with and without in-surface inertia terms, for a closed circular shell vibrating in the first and third longitudinal modes, $m = 1$ and 3.

For this set of curves, the class of shells had the following properties:

$$a/L_x = \tfrac{1}{4},\ \mu = 0.3,\ h/L_x = 0.008333$$

The value of p was taken as 25. For higher circumferential modes n_c the value of p could be reduced and still obtain accurate values for the frequencies. For the first mode shape in the η-direction ($m = 1$), all natural frequencies were accurately calculated for the second through the ninth circumferential modes of closed circular cylinder ($n_c = 2$ to 9). For $m = 3$, the natural frequencies for circumferential modes 4 to 9 were obtained with reasonable accuracy. For the second circumferential mode, neglecting in-surface inertias, the calculated frequency was 19% below that given by the analysis of Stadler and Wang. For the third circumferential mode, the error was only 15%. If more terms were taken, this error would decrease for these lower circumferential modes. These results verify those given earlier for cases in which p was fixed. This indicates that the present method is limited practically to open panels or to the calculation of higher natural frequencies for closed oval shells. For these shell parameters, the method gives good results for circular panels having the value L_s/r less than 1.5. Better accuracy was obtained for the first longitudinal mode ($m = 1$) than the third longitudinal mode ($m = 3$).

For lower circumferential modes, solutions which neglect in-surface inertia terms give higher natural frequencies than corresponding solutions which include in-surface inertia. However, for the higher circumferential modes, in-surface inertias do not appreciably affect the natural frequencies. These results have been observed by Armenakas,[13] Ivanuta and Finkel'shteyn,[14] and many others.[11]

The final study made for circular cylindrical shell panels used the following boundary conditions at constant ξ:

$$u = v = w = M_\xi = 0,\ \text{at } \xi = 0,1$$

For a circular cylindrical panel with this set of boundary conditions, the first mode shape (a predominately stretching mode) occurred at a much higher natural frequency. This behavior is different for circular panels extracted from vibrating circular cylindrical shells. As previously noted, the straight edges of panels extracted from vibrating circular shells are allowed to translate in the plane of the surface during vibration, and the lowest frequency coincides with the fundamental mode shape. When the straight edges are not allowed to translate, the lowest frequency coincides with the second circumferential mode.

In-surface inertia terms were neglected, and the panel was allowed to vibrate in the first longitudinal mode. The lowest vibrational frequencies are shown in Fig. 2.

For all practical purposes, there was very little experimental or analytical data with which to compare natural frequencies of noncircular cylinders. Therefore, it was decided to compare the natural frequencies of two noncircular cylinders that were mirror images of each other. One would expect these two cylinders to vibrate with the same natural frequencies and that the D_j's for the nondimensional curvature, while being numerically different, would in reality describe the same shell. This objective was accomplished by using the following expressions for the nondimensional curvature:

$$L_s/r = \pi/4 + \pi\xi/16 \text{ and } L_s/r = 5\pi/16 - \pi\xi/16$$

It was found that the calculated natural frequencies were exactly the same for both cylinders. The fundamental nondimensional frequency was $\bar{\omega}_1{}^2 = 0.03207$, and the second nondimensional frequency was $\bar{\omega}_2{}^2 = 0.04243$.

Because only characteristic shapes are calculated for free vibrational problems, the exact values of the displacements were not found. Instead, the displacements were redefined as follows:

$$\bar{u} = u/w_{\max},\ \bar{v} = v/w_{\max},\ \bar{w} = w/w_{\max}$$

The modes for the first two natural frequencies are shown in Figs. 3 and 4. From Eq. (5) and setting $m = 1$, the maximum $\bar{v}$ and $\bar{w}$ displacements occur at $\eta = \frac{1}{4}$ and the maximum value of $\bar{u}$ occurs at $\eta = 0$.

The maximum w displacement occurred at approximately $\xi = 0.43$ for the lowest natural frequency, as shown in Fig. 3. The $\bar{v}$ displacement has approximately the same magnitude, but different signs, at $\xi = 0$ and 1. The $\bar{u}$ displacement had a similar shape to the $\bar{w}$ displacement, but a much smaller magnitude. Because there was no available information on the modes of free vibration of noncircular cylinders, only a qualitative analysis of these displacements could be made. From the analysis of circular cylindrical panels with the same boundary conditions, the displacements appear to be quite reasonable.

The normalized displacements for the second natural frequency, shown in Fig. 4, seem reasonable. Boyd[12] studied a similar noncircular cylinder that was statically loaded by uniform pressure, and obtained a w-deflection curve similar in shape to that shown for the second frequency mode. In both cases, the maximum displacements occurred when ξ was greater than 0.5. For circular panels vibrating in the second mode, the $\bar{w}$ displacements would be equal at $\xi = 0.25$ and 0.75.

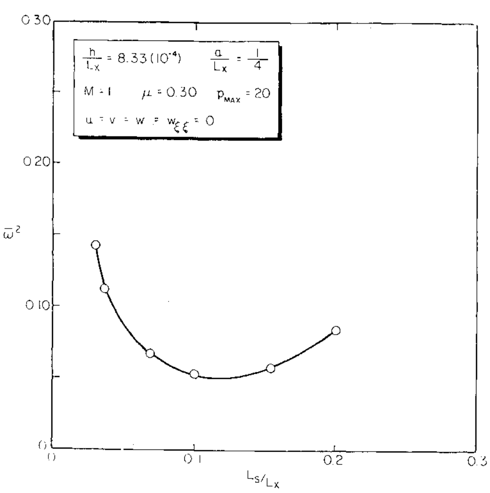


Fig. 2 Lowest nondimensional natural frequency (second mode shape), $m = 1$; in-surface inertia neglected.

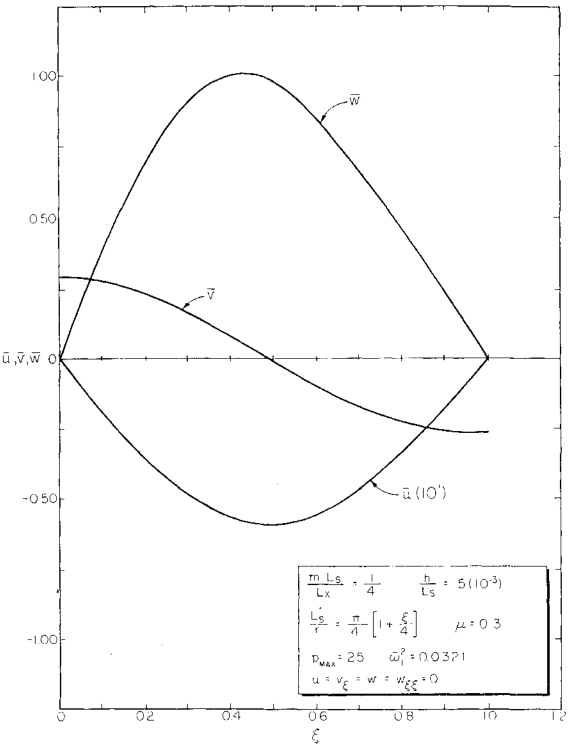


Fig. 3 Normalized displacements for first circumferential mode, $m = 1$; in-surface inertia neglected.

Now that the power series method was shown to give the natural frequencies for noncircular cylindrical panels, the frequencies of a family of noncircular panels were calculated. The curvature expression for this family of noncircular cylindrical panels is given by the expression

$$L_s/r = \pi/4 + b\xi \tag{10}$$

where b is an arbitrary constant.

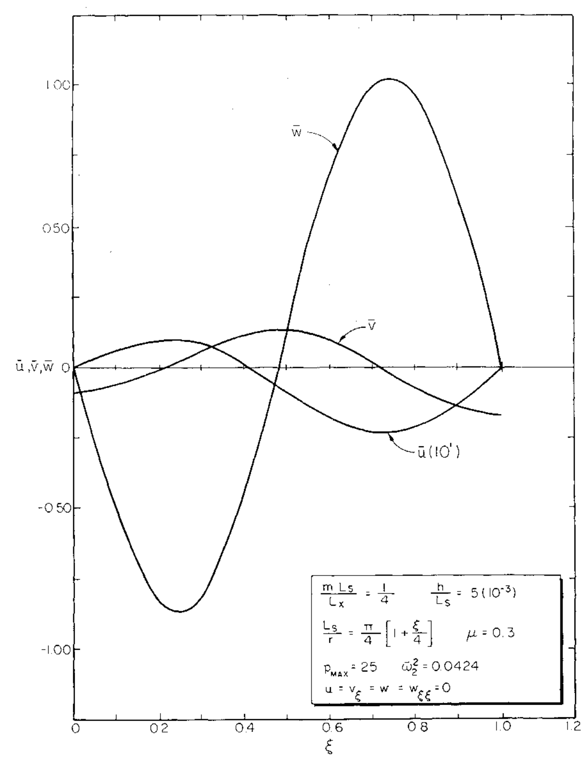
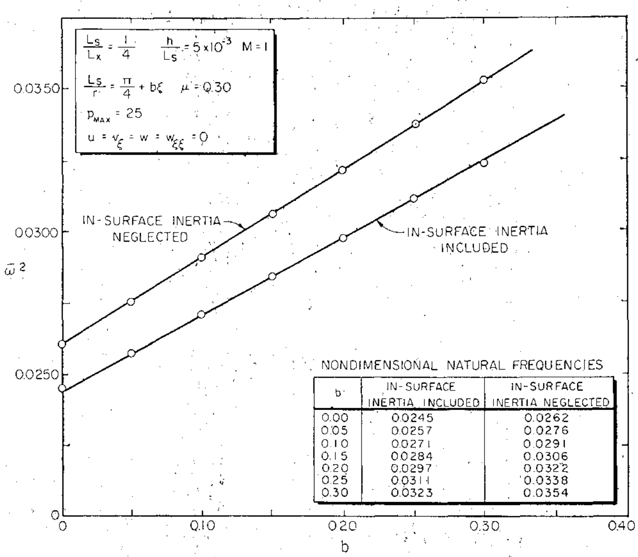


Fig. 4 Normalized displacements for second circumferential mode, $m = 1$; in-surface inertia neglected.



b	IN-SURFACE INERTIA INCLUDED	IN-SURFACE INERTIA NEGLECTED
0.00	0.0245	0.0262
0.05	0.0257	0.0276
0.10	0.0271	0.0291
0.15	0.0284	0.0306
0.20	0.0297	0.0322
0.25	0.0311	0.0338
0.30	0.0323	0.0354

Fig. 5 Lowest nondimensional frequency vs b, $m = 1$; in-surface inertia included and neglected.

The value of b was allowed to vary from 0.0 to 0.3. The boundary conditions along the straight edges of the panel were

$$u = v_\xi = w = M_\xi = 0 \text{ at } \xi = 0,1$$

The natural frequency was nearly linear with b, with or without in-surface inertias included as shown in Fig. 5. Numerically, the two lines diverged for higher values of b, but the ratios of corresponding frequencies converged. Higher natural frequencies were obtained when in-surface inertias were neglected. A similar result was obtained for circular cylinders.

Equivalent circular cylindrical panels, which vibrate at the same natural frequencies as the corresponding noncircular panel, were found. The curvature of the equivalent circular panel was found by the expression

$$L_s/r = \pi/4 + b\xi_{\text{equiv}}$$

For each value of b, the value of ξ_{equiv} was found to be approximately 0.47.

The frequency and mode shape for the second circumferential mode were calculated for only one value of b. The value of this frequency and the corresponding mode shape appeared to be quite reasonable. Its shape was similar to that shown by Fig. 4.

Conclusions

Calculations of the natural frequencies and modes of free vibration of circular and noncircular cylindrical panels were made using the power series method. The method provided better accuracy for thinner shell panels and for shell panels with higher length-to-radius ratios. This power series method is most applicable to the class of noncircular cylindrical panels whose curvature is easily expressed as a power series. A significant advantage of this method is that the natural frequencies of cylindrical panels can be obtained for all combinations of straight-edge boundary conditions without having to reformulate the problem for each set of boundary conditions.

For circular cylindrical panels having simply supported straight-edge boundary conditions, the in-surface displacement boundary conditions significantly affected the natural frequencies. When the straight edges were not allowed to translate in the plane of the shell, the natural frequencies were an order of magnitude higher than when the straight edges are allowed in-surface translation. Also, the modes corresponding to the lowest frequencies were of different shapes. With proper choice of boundary conditions, the natural frequencies of closed circular cylindrical shells can be found, but there is no apparent advantage of the power series method over other methods.

Higher natural frequencies were calculated when in-surface inertia terms were neglected for noncircular cylindrical panels. This is the same effect found in the analysis of circular cylindrical panels.

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